First Quarterly Report

MHD BOUNDARY LAYERS INVOLVING NON-EQUILIBRIUM IONIZATION

(25 January 1967 - 24 April 1967)

by

Arthur Sherman
Eli Reshotko (Consultant)
Giles Crimi
Frank Cristinzio

Prepared under Contract No. NASw-1586 by
Space Sciences Laboratory
GENERAL ELECTRIC COMPANY
King of Prussia, Pa.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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I. INTRODUCTION

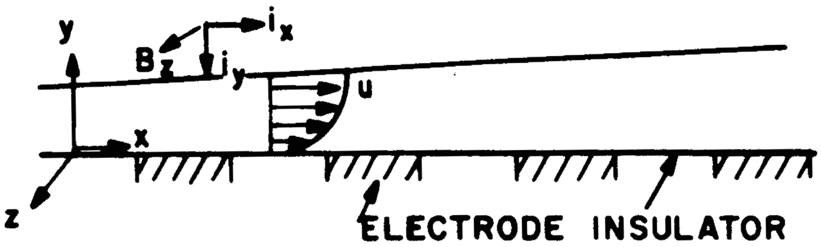
The present report is the first quarterly report under Contract NASw-1586 and covers the period 25 January 1967 to 24 April 1967. The problem being considered is the boundary layer that will develop over the segmented electrode wall in an MHD Channel when a non-equilibrium plasma is flowing. In our analysis the characteristics of the plasma sheath at the base of the continuum boundary layer as well as the variation of the electron temperature through out the boundary layer are considered. Due to our desire to study finite electrode segments in the wall over which the boundary layer develops, we will use a finite difference technique in solving the boundary layer equations. This will permit the study of non-similar behavior in a reasonably accurate way.

II. ANALYSIS

The formulation of our problem will be for a two temperature plasma under the following simplifying assumptions

1. Steady flow $\frac{\partial}{\partial t} = 0$
2. Laminar flow
3. No induced magnetic fields $R_m \cong 0$
4. Plasma consists only of electrons, atoms (carrier and seed), singly ionized ions (carrier and seed).
5. Plasma composition determined by Saha equation evaluated at the electron temperature.
6. No continuum radiation losses
7. Collision free plasma sheath
8. Only thermionic emission
9. Neglect pressure differences normal to wall

The geometry of wall along which the boundary layer will develop is shown below.



The basic equations are developed below in boundary layer form.

Mass Conservation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

Momentum Conservation:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + j_y B_z \tag{2}$$

Overall energy conservation:

$$\rho u \frac{\partial h^*}{\partial x} + \rho v \frac{\partial h^*}{\partial y} = u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial}{\partial y} (q_y)$$

$$+ j_x E_x + j_y (E_y - uB)$$

The heat flux vector is assumed to be of the following form

$$\underset{\sim}{q} = \sum_s \underset{\sim}{q}_s \quad \text{where} \quad \underset{\sim}{q}_s = - K_s \nabla T_s + \rho_s h_s \underset{\sim}{V}_s$$

Now let us express h^* more explicitely

$$h^* = \frac{1}{\rho} \sum_s \rho_s h_s^* \qquad h_s^* = \frac{5}{2} \frac{kT_s}{m_s} + \frac{I_s}{m_s}$$

so that

$$h^* = \left(\frac{\rho_A}{\rho} \frac{5}{2} \frac{kT}{m_A} + \frac{\rho_{C_s}}{\rho} \frac{5}{2} \frac{kT}{m_{C_s}} + \frac{\rho_{A^+}}{\rho} \frac{5}{2} \frac{kT}{m_{A^+}} \right.$$

$$\left. + \frac{\rho_{C_s^+}}{\rho} \frac{5}{2} \frac{kT}{m_{C_s}} + \frac{\rho_e}{\rho} \frac{5}{2} \frac{kT_e}{m_e} \right) + \frac{\rho_{A^+}}{\rho} \frac{I_A}{m_A} + \frac{\rho_{C_s^+}}{\rho} \frac{I_{C_s}}{m_{C_s}}$$

or

$$h^* = h + \frac{n_{A^+}}{\rho} I_A + \frac{n_{C_s^+}}{\rho} I_{C_s}$$

We can then rewrite the heat flux vector for s as follows

$$\underset{\sim}{q}_s = - K_s \nabla T_s + m_s n_s h_s \underset{\sim}{V}_s + \frac{m_s n_s^2}{\rho} I_s \underset{\sim}{V}_s$$

so that

$$\underset{\sim}{q} = - \sum_s K_s \nabla T_s + m_e n_e \frac{5}{2} \frac{kT_e}{m_e} \underset{\sim}{V}_e = - \sum_s K_s \nabla T_s - \frac{5kT_e}{2e} \underset{\sim}{j}_e$$

Then the overall energy equation becomes

$$\rho u \frac{\partial h^*}{\partial x} + \rho v \frac{\partial h^*}{\partial y} = u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y} \left(\sum_s K_s \frac{\partial T_s}{\partial y} + \frac{5kT_e}{2e} j_{e_y} \right)$$

$$+ j_x E_x + j_y (E_y - uB)$$

$$- \rho u I_A \frac{\partial}{\partial x} \left(\frac{n_{A^+}}{\rho}\right) - \rho u I_{C_s} \frac{\partial}{\partial x} \left(\frac{n_{C_s^+}}{\rho}\right)$$

$$- \rho v I_A \frac{\partial}{\partial y} \left(\frac{n_{A^+}}{\rho}\right) - \rho v I_{C_s} \frac{\partial}{\partial y} \left(\frac{n_{C_s^+}}{\rho}\right)$$

Next, let us write

$$\rho u I_{C_s} \frac{\partial}{\partial x} \left(\frac{n_{C_s^+}}{\rho}\right) + \rho v I_{C_s} \frac{\partial}{\partial y} \left(\frac{n_{C_s^+}}{\rho}\right)$$

$$= I_{C_s} \left\{ u \frac{\partial n_{C_s^+}}{\partial x} + v \frac{\partial n_{C_s^+}}{\partial y} - n_{C_s^+} \left(\frac{u}{\rho} \frac{\partial \rho}{\partial x} + \frac{v}{\rho} \frac{\partial \rho}{\partial y} \right) \right\}$$

and the energy equation can be rewritten as

$$\rho u \frac{\partial h^*}{\partial x} + \rho v \frac{\partial h^*}{\partial y} = u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y} \left(\sum_s K_s \frac{\partial T_s}{\partial y} + \frac{5kT_e}{2e} j_{e_y} \right)$$

$$+ j_x E_x + j_y (E_y - uB)$$

$$- I_{c_s} \left[u \frac{\partial n_{C_s^+}}{\partial x} + v \frac{\partial n_{C_s^+}}{\partial y} - n_{C_s} \left(\frac{u}{\rho} \frac{\partial \rho}{\partial x} + \frac{v}{\rho} \frac{\partial \rho}{\partial y} \right) \right]$$

$$- I_A \left[u \frac{\partial n_{A^+}}{\partial x} + v \frac{\partial n_{A^+}}{\partial y} - n_{A^+} \left(\frac{u}{\rho} \frac{\partial \rho}{\partial x} + \frac{v}{\rho} \frac{\partial \rho}{\partial y} \right) \right]$$

and the last two terms are equivalent to the $\sum_{\text{ions}} \dot{\omega}_s I_s$ term commonly included in the energy equation.

Next $\frac{\partial p}{\partial x}$ can be obtained from the momentum equation evaluated at the free stream. Then

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) + j_y B_z$$

and at ∞

$$\rho_\infty u_\infty \frac{du_\infty}{dx} = -\frac{\partial p}{\partial x} + j_y B_z$$

$$\therefore \quad \frac{\partial p}{\partial x} = j_y B_z - \rho_\infty u_\infty \frac{du_\infty}{dx}$$

and the energy equation becomes

$$\rho u \frac{\partial h^*}{\partial x} + \rho v \frac{\partial h^*}{\partial y} = -\rho_\infty u_\infty u \frac{du_\infty}{dx} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\sum_s K_s \frac{\partial T_s}{\partial y} + \frac{5kT_e}{2e} j_{e_y}\right)$$

$$+ j_x E_x + j_y E_y$$

$$- I_{C_s}\left[u \frac{\partial n_{C_s^+}}{\partial x} + v \frac{\partial n_{C_s^+}}{\partial y} - n_{C_s^+}\left(\frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial y}\right)\right]$$

$$- I_A\left[u \frac{\partial n_{A^+}}{\partial x} + v \frac{\partial n_{A^+}}{\partial y} - n_{A^+}\left(\frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial y}\right)\right]$$

Finally, we have to reexpress the two last terms on the RHS in terms of T_e, T, u. Now $n_{C_s^+}$ and n_{A^+} can be obtained from the Saha relation.

$$\frac{n_e n_{C_s^+}}{n_{C_s}} = \left(\frac{2\pi m_e k T_e}{k^2}\right)^{3/2} \exp\left[-\frac{e I_{C_s}}{kT_e}\right] = S(T_e)$$

$$\frac{n_e n_{A^+}}{n_A} = 12\left(\frac{2\pi m_e k T_e}{k^2}\right)^{3/2} \exp\left[-\frac{e I_A}{kT_e}\right] = r\,(T_e)$$

also $n_e = n_{C_s^+} + n_{A^+}$. The ratio of the original number density of C_s as compared to Argon will be specified. So

$$P \equiv \frac{n_{C_s^+} + n_{C_s}}{n_{A^+} + n_A} \simeq \frac{n_{C_s^+} + n_{C_s}}{n_A}$$

Also, assuming each species is a P.G.

$$p = \sum_i p_i = k\left[\, n_A + n_{C_s} - n_A + n_{C_s^+}\right] T + k\, n_e T_e$$

or $\quad p \cong k\, n_A T$

Now we write from Saha

$$\left(n_{C_s^+}^2\right) + \left(n_{C_s^+}\right)\left(n_{A^+}\right) = S(T_e)\, n_{C_s}$$

$$\left(n_{A^+}^2\right) + \left(n_{C_s^+}\right)\left(n_{A^+}\right) = r\,(T_e)\, n_A$$

Assuming, however, $n_{C_s^+} \cong 10^{14}/CC$ $n_A \cong 10^{12}/CC$ then

$$n_{C_s^+}^2 \cong S(T_e)\, n_{C_s}$$

and

$$\left(n_{C_s^+}\right)\left(n_{A^+}\right) = r\,(T_e)\, n_A$$

But from the definition of P

$$n_{C_s} = P_{n_A} - n_{C_s^+}$$

$$\therefore \quad n_{C_s^+}^2 = S(T_e)\left[P\frac{p}{kT} - n_{C_s^+}\right]$$

or $$n_{C_s^+}^2 + S\,(T_e)\, n_{C_s^+} - \frac{PpS}{kT} = 0$$

and $$n_{C_s^+} = -\frac{S}{2} + \sqrt{\frac{S^2}{4} + \frac{PpS}{kT}}$$

or $$n_{C_s^+} = \frac{S}{2}\left\{\sqrt{1 + \frac{4Pp}{kTS}} - 1\right\} \quad \text{when } \frac{4Pp}{kTS} \geq 10^{-2}$$

When S is very large, corresponding to nearly full ionization, the above may prove very inaccurate for numerical calculation. For this case we expand the $\sqrt{}$ and use

$$n_{C_s^+} = \frac{pP}{kT}\left[1 - \frac{Pp}{kTS}\right] \quad \text{when } \frac{4Pp}{kTS} < 10^{-2}$$

also

$$n_{A^+} = \frac{r(T_e)\, n_A}{n_{C_s^+}}$$

If we wish we can also write

$$S(T_e) = C_1 \, T_e^{3/2} \, e^{-C_2/T_e}$$

$$r(T_e) = 12 \, C_1 \, T_e^{3/2} \, e^{-C_3/T_e}$$

where

$$C_1 = \left(\frac{2\pi m_e k T_e}{k^2} \right)^{3/2}$$

$$C_2 = eI_{C_s}/k \qquad C_3 = eI_A/k$$

Then we can write out to begin with $\dfrac{\partial n_{C_s^+}}{\partial x}$ using

$$n_{C_s^+} = -\frac{S}{2} + \sqrt{\frac{S^2}{4} + \frac{PpS}{kT}}$$

and $S(T_e)$ directly above.

Now

$$\frac{\partial n_{C_s^+}}{\partial x} = S_1 \frac{\partial T_e}{\partial x} - S_2 \frac{\partial h}{\partial x} + S_3 \frac{dp}{dx}$$

where

$$S_1 = S \left(\frac{3}{2T_e} + \frac{C_2}{T_e^2} \right) \left[-\frac{1}{2} + \frac{\frac{S}{2} + \frac{Pp}{kt}}{S\sqrt{1 + \frac{4Pp}{kTS}}} \right]$$

$$S_2 = \frac{4Pp}{kC_p T^2 \sqrt{1 + \frac{4Pp}{kTS}}} \qquad S_3 = \frac{4P}{kt\sqrt{1 + \frac{4Pp}{kTS}}}$$

Similarly

$$\frac{\partial n_{C_s^+}}{\partial y} = S_1 \frac{\partial T_e}{\partial y} - S_2 \frac{\partial h}{\partial y} \qquad \left(\frac{\partial p}{\partial y} = 0 \right)$$

Accordingly, neglecting Argon ionization the overall energy equation becomes

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = - \rho_\infty u_\infty u \frac{du_\infty}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2 \left(\sum_s K_s \frac{\partial T_s}{\partial y} + \frac{5k T_e}{2e} \quad j_{e_y}\right)$$

$$+ j_x E_x + j_y E_y$$

$$- I_{C_s} \left[u S_1 \frac{\partial T_e}{\partial x} - u S_2 \frac{\partial h}{\partial x} + u S_3 \frac{dp}{dx} \right.$$

$$\left. + v S_1 \frac{\partial T_e}{\partial y} - v S_2 \frac{\partial h}{\partial y} - n_e \left(\frac{u}{\rho} \frac{\partial \rho}{\partial x} + \frac{v}{\rho} \frac{\partial \rho}{\partial y}\right)\right]$$

but the last two terms can be rewritten as follows: Assume the overall plasma density, pressure, et al, is that of a perfect gas (argon). Then

$$p = \rho R T = \rho \frac{R}{C_p} h \quad \therefore \quad \rho = \frac{C_p}{R} \frac{p}{h}$$

Then

$$\frac{1}{\rho} \frac{\partial \rho}{\partial x} = - \frac{1}{\rho} \frac{C_p}{R} p \frac{1}{h^2} \frac{\partial h}{\partial x} + \frac{1}{\rho} \frac{C_p}{R} \frac{1}{h} \frac{dp}{dx}$$

$$= - \frac{1}{h} \frac{\partial h}{\partial x} + \frac{1}{p} \frac{dp}{dx}$$

$$\therefore \quad \frac{u}{\rho} \frac{\partial \rho}{\partial x} + \frac{v}{\rho} \frac{\partial \rho}{\partial y} = - \frac{u}{h} \frac{\partial h}{\partial x} - \frac{v}{h} \frac{\partial h}{\partial y} + \frac{u}{p} \frac{dp}{dx}$$

The overall energy equation is then

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = - \rho_\infty u_\infty u \frac{du_\infty}{dx} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\sum_s K_s \frac{\partial T_s}{\partial y} + \frac{5kT_e}{2e} j_{e_y}\right)$$

$$+ j_x E_x + j_y E_y - I_{C_s} \left[u S_1 \frac{\partial T_e}{\partial x} - u S_2 \frac{\partial h}{\partial x} + u S_3 \frac{dp}{dx} \right.$$

$$\left. + v S_1 \frac{\partial T_e}{\partial y} - v S_2 \frac{\partial h}{\partial y} - n_e \left(\frac{u}{p} \frac{dp}{dx} - \frac{u}{h} \frac{\partial h}{\partial x} - \frac{v}{h} \frac{\partial h}{\partial y}\right)\right]$$

But we know $\frac{dp}{dx}$ to be $= j_y B_z - \rho_\infty u_\infty \frac{du_\infty}{dx}$. Then

$$
\begin{aligned}
\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = & - \rho_\infty u_\infty \frac{du_\infty}{dx}\, u + \mu \left(\frac{\partial u}{\partial y}\right)^2 \\
& + \frac{\partial}{\partial y}\left(\sum_s K_s \frac{\partial T_s}{\partial y} + \frac{5kT_e}{2e}\, j_{e_y} \right) \\
& + j_x E_x + j_y E_y - I_{C_s} \Bigg[u S_1 \frac{\partial T_e}{\partial x} - u S_2 \frac{\partial h}{\partial x} \\
& + u S_3\, j_y B_z - u S_3 \rho_\infty u_\infty \frac{du_\infty}{dx} + v S_1 \frac{\partial T_e}{\partial y} - v S_2 \frac{\partial h}{\partial y} \\
& - n_e \left(\frac{u}{p}\, j_y B_z - \frac{u}{p}\, \rho_\infty u_\infty \frac{du_\infty}{dx} - \frac{u}{h} \frac{\partial h}{\partial x} - \frac{v}{h} \frac{\partial h}{\partial y} \right) \Bigg]
\end{aligned}
$$

Before proceeding it now is clear that we have to say something about j_x, E_x, j_y, E_y. We will make the following assumption.

1. Over an electrode $j_y = j_{y_\infty}(x)$, $E_x = 0$ for all y's.
2. Over an insulator $j_y = 0$, $E_x = E_{x_\infty}(x)$ for all y's.

Then j_x and E_y can be determined from the two components of Ohms Law. Neglecting electron pressure gradients they are

$$j_x = \frac{\sigma}{(1+\beta_e\beta_i)^2 + \beta_e^2} \left\{ (1+\beta_e\beta_i)\, E_{x_\infty} - \beta_e\, (E_y - uB_z) \right\}$$

$$j_{y_\infty} = \frac{\sigma}{(1+\beta_e\beta_i)^2 + \beta_e^2} \left\{ (1+\beta_e\beta_i)\,(E_y - uB_z) + \beta_e E_{x_\infty} \right\}$$

use the 2nd relation to replace $(E_y - uB_z)$ in 1st.

we get

$$j_x = \frac{\sigma}{1+\beta_e\beta_i} \left\{ E_{x_\infty} - \frac{\beta_e}{\sigma}\, j_{y_\infty} \right\}$$

and

$$j_x E_x = \frac{\sigma}{1+\beta_e\beta_i} E_{x_\infty}^2 - \underbrace{\frac{\beta_e}{1+\beta_e\beta_i}\, j_{y_\infty} E_{x_\infty}}_{\to 0} \qquad \left\{ \begin{array}{l} j_{y_\infty} = 0 \text{ where } E_{x_\infty} \neq 0 \\ E_{x_\infty} = 0 \text{ where } j_{y_\infty} \neq 0 \text{ .} \end{array} \right.$$

So

$$j_x E_x = \frac{\sigma}{1+\beta_e \beta_i} E_{x_\infty}^2$$

Next, solve the 2nd of above eqs. for E_y.

$$E_y = \frac{j_y \left[(1+\beta_e\beta_i)^2 + \beta_e^2\right]}{\sigma(1+\beta_e\beta_i)} - \frac{\beta_e}{1+\beta_e\beta_i} E_{x_\infty} + u B_z$$

Then

$$j_y E_y = \frac{j_{y_\infty}^2}{\sigma} \left[\frac{(1+\beta_e\beta_i)^2 + \beta_e^2}{1+\beta_e\beta_i}\right] + u B_z j_{y_\infty}$$

Using these relations the overall energy equation becomes

$$\rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} = -\left(\rho_\infty u_\infty \frac{du_\infty}{dx}\right) u + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left[\sum_s K_s \frac{\partial T_s}{\partial y}\right.$$
$$\left. + \frac{5kT_e}{2e}\left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}\right)\right] + \frac{\sigma}{1+\beta_e\beta_i} E_{x_\infty}^2$$
$$+ \frac{j_{y_\infty}^2}{\sigma}\left[\frac{(1+\beta_e\beta_i)^2+\beta_e^2}{1+\beta_e\beta_i}\right] + u B_z j_{y_\infty}$$
$$- I_{C_s}\left[u S_1 \frac{\partial T_e}{\partial x} - u S_2 \frac{\partial h}{\partial x} + u S_3 j_{y_\infty} B_z - u S_3 \rho_\infty u_\infty \frac{du_\infty}{dx}\right.$$
$$+ v S_1 \frac{\partial T_e}{\partial y} - v S_2 \frac{\partial h}{\partial y} - n_e\left(\frac{u}{p} j_{y_\infty} B_z - \frac{u}{p} \rho_\infty u_\infty \frac{du_\infty}{dx}\right.$$
$$\left.\left. - \frac{u}{h}\frac{\partial h}{\partial x} - \frac{v}{h}\frac{\partial h}{\partial y}\right)\right]$$

Electron energy Conservation:

The general form will be taken to be

$$\nabla \cdot \left(n_e \underset{\sim}{v}\left[\frac{3}{2}kT_e + I_{C_s}\right]\right) + \nabla \cdot \left[-K_e \nabla T_e - \frac{5}{2}\frac{k}{e} T_e \underset{\sim}{j}_e - \frac{I_{C_s}}{e} \underset{\sim}{j}_e\right]$$
$$+ p_e \nabla \cdot \underset{\sim}{v} = \underset{\sim}{E}^* \cdot \underset{\sim}{j}_e + 3\rho_e k (T-T_e) \sum_s \nu_{e_s}^* / m_s$$

which we can rewrite as

$$\nabla\cdot\left(n_e\underset{\sim}{v}\left[\frac{3}{2}\ kT_e\right]\right) - \nabla\cdot\left[K_e\nabla T_e + \frac{5}{2}\ \frac{k}{e}\ T_e\, \underset{\sim}{j}_e\right] + p_e\nabla\cdot\underset{\sim}{v}$$

$$= \underset{\sim}{E}^*\cdot\underset{\sim}{j}_e + 3\rho_e\, k\,(T-T_e)\,\Sigma\,\nu_{e_s}/m_s + \frac{I_{C_s}}{e}\ \nabla\cdot\underset{\sim}{j}_e$$

$$- I_{C_s}\nabla\cdot(n_e\underset{\sim}{v})$$

In boundary layer form this can be simplified to

$$\frac{3}{2}\,k\,u\,n_e\frac{\partial T_e}{\partial x} + \frac{3}{2}\,k\,u\,T_e\ \frac{\partial n_e}{\partial x} + \frac{3}{2}kv\,n_e\frac{\partial T_e}{\partial y} + \frac{3}{2}\,k\,v\,T_e\frac{\partial n_e}{\partial y}$$

$$+ n_e\left[\frac{5}{2}\,kT_e + I_{C_s}\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left[K_e\,\frac{\partial T_e}{\partial y} + \frac{5}{2}\,\frac{k}{e}\,T_e\, j_{e_y}\right]$$

$$= E_x\, j_{e_x} + (E_y - uB_z)\, j_{e_y} + 3\,\rho_e\,k\,(T-T_e)\,\Sigma\ \frac{\nu_{e_s}}{n_s}$$

$$- I_{C_s}\,u\ \frac{\partial n_e}{\partial x}\ - I_{C_s}\ v\ \frac{\partial n_e}{\partial y}$$

Now we have as before

$$\frac{\partial n_e}{\partial x} = S_1\ \frac{\partial T_e}{\partial x}\ - S_2\ \frac{\partial h}{\partial x} + S_3\ \frac{dp}{dx}$$

$$\frac{\partial n_e}{\partial y} = S_1\ \frac{\partial T_e}{\partial y}\ - S_2\ \frac{\partial h}{\partial y}$$

Now we need an expression for j_{e_x} and j_{e_y}. This we obtain from

$$\underset{\sim}{j}_e = \underset{\sim}{j} + \beta_i\ \frac{\underset{\sim}{B}\times\underset{\sim}{j}}{B_z}$$

Then

$$j_{e_x} = j_x - \frac{\beta_i}{B_z}\ B_z\, j_y = j_x - \beta_i\, j_y$$

$$j_{e_y} = j_y + \frac{\beta_i B_z}{B_z}\ j_x = j_y + \beta_i\, j_x$$

also as before

$$j_x = \frac{\sigma}{1+\beta_e\beta_i}\left[E_{x_\infty} - \frac{\beta_e}{\sigma} j_{y_\infty}\right]$$

So that

$$j_{e_y} = j_{y_\infty}\left[\frac{1}{1+\beta_e\beta_i}\right] + \left(\frac{\sigma\beta_i}{1+\beta_e\beta_i}\right) E_{x_\infty}$$

or

$$j_{e_y} = \alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}$$

then

$$E_{x_\infty} j_{e_x} = E_{x_\infty} j_x - \beta_i j_{y_\infty} \Big|_{E_{x_\infty} \to 0}$$

or

$$E_{x_\infty} j_{e_x} = \frac{\sigma}{1+\beta_e\beta_i} E_{x_\infty}^2$$

and

$$(E_y - uB) j_{e_y} = \frac{\left[(1+\beta_e\beta_i)^2 + \beta_e^2\right]}{\sigma(1+\beta_e\beta_i)^2} \cdot j_{y_\infty}^2 - \frac{\beta_e}{(1+\beta_e\beta_i)^2} \cdot \sigma\beta_i E_{x_\infty}^2$$

then

$$j_{e_x} E_{x_\infty} + j_{e_y}(E_y - uB_z) = \left[\frac{(1+\beta_e\beta_i)^2 + \beta_e^2}{(1+\beta_e\beta_i)^2}\right]\frac{j_{y_\infty}^2}{\sigma} + \frac{\sigma}{(1+\beta_e\beta_i)^2} E_{x_\infty}^2$$

So that the electron energy equation can be rewritten

$$\frac{3}{2} k u n_e \frac{\partial T_e}{\partial x} + \frac{3}{2} k v n_e \frac{\partial T_e}{\partial y} + \left(\frac{3}{2} k T_e + I_{C_s}\right) u \left\{S_1 \frac{\partial T_e}{\partial x} - S_2 \frac{\partial h}{\partial x} + S_3 \frac{dp}{dx}\right\}$$

$$+ \left(\frac{3}{2} kT_e + I_{C_s}\right) v \left\{S_1 \frac{\partial T_e}{\partial y} - S_2 \frac{\partial h}{\partial y}\right\} + n_e\left[\frac{5}{2} kT_e + I_{C_s}\right]\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$- \frac{\partial}{\partial y}\left[K_e \frac{\partial T_e}{\partial y} + \frac{5}{2}\frac{k}{e} T_e \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}\right)\right] = \left[\frac{(1+\beta_e\beta_i)^2 + \beta_e^2}{(1+\beta_e\beta_i)^2}\right]\frac{j_{y_\infty}^2}{\sigma}$$

$$+ \frac{\sigma}{(1+\beta_e\beta_i)^2} E_{x_\infty}^2 + 3 m_e n_e k (T - T_e) \Sigma \frac{\nu_{e_s}}{m_s}$$

Now we have to reexpress $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$. Thus

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

and

$$\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{u}{\rho}\frac{\partial \rho}{\partial x} - \frac{v}{\rho}\frac{\partial \rho}{\partial y} = \rho\left[u\frac{\partial \rho^{-1}}{\partial x} + v\frac{\partial \rho^{-1}}{\partial y}\right]$$

and we as well replace $\frac{dp}{dx}$ from before

$$\frac{dp}{dx} = j_{y_\infty} B_z - \rho_\infty u_\infty \frac{du_\infty}{dx}$$

The electron energy equation then becomes

$$\frac{3}{2} kun_e \frac{\partial T_e}{\partial x} + \frac{3}{2} kvn_e \frac{\partial T_e}{\partial y} + \left(\frac{3}{2} kT_e + I_{C_s}\right) u \left\{ S_1 \frac{\partial T_e}{\partial x} - S_2 \frac{\partial h}{\partial x}\right.$$

$$\left. + S_3 j_{y_\infty} B_z - S_3 \rho_\infty u_\infty \frac{du_\infty}{dx}\right\} + \left(\frac{3}{2} kT_e + I_{C_s}\right) v \left\{ S_1 \frac{\partial T_e}{\partial y} - S_2 \frac{\partial h}{\partial y}\right\}$$

$$+ n_e\left[\frac{5}{2} kT_e + I_{C_s}\right]\left\{\rho u \frac{\partial \rho^{-1}}{\partial x} + \rho v \frac{\partial \rho^{-1}}{\partial y}\right\} - \frac{\partial}{\partial y}\left[K_e \frac{\partial T_e}{\partial y}\right.$$

$$\left. + \frac{5}{2}\frac{k}{e} T_e \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}\right)\right] = \left[\frac{(1+\beta_e\beta_i)^2 + \beta_e^2}{(1+\beta_e\beta_i)^2}\right] \frac{j_{y_\infty}^2}{\sigma}$$

$$+ \frac{\sigma}{(1+\beta_e\beta_i)^2} E_{x_\infty}^2 + 3m_e n_e k (T - T_e) \sum_s \frac{\nu_{e_s}}{m_s}$$

We next transform these four equations to new independent variables, those of the Levy-Lees transformation, as follows

$$\xi(x) = \int_0^x (\rho\mu)_r u_\infty \, dx$$

$$\eta(x,y) = \frac{u_\infty}{\sqrt{2\xi}} \int_0^y \rho \, dy$$

so that

$$\frac{\partial}{\partial x} = (\rho\mu)_r u_\infty \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = \frac{\rho u_\infty}{\sqrt{2\xi}} \frac{\partial}{\partial \eta}$$

and where

$$V \equiv \frac{2\xi}{(\rho\mu)_r u_\infty} \left(f' \eta_x + \frac{\rho v}{\sqrt{2\xi}} \right)$$

The equations become

Continuity:

$$2\xi \frac{\partial f'}{\partial \xi} + \frac{\partial V}{\partial \eta} + f' = 0$$

Momentum:

$$2\xi f' \frac{\partial f}{\partial \xi} + V \frac{\partial f'}{\partial \eta} = \frac{2\xi}{u_\infty} \frac{du_\infty}{d\xi} \left[g - f'^2 \right] + \frac{\partial}{\partial \eta} \left(\ell \frac{\partial f'}{\partial \eta} \right)$$

Energy:

$$2\xi f' \frac{\partial g}{\partial \xi} + V \frac{\partial g}{\partial \eta} + \frac{2\xi f'g}{h_\infty} \frac{dh_\infty}{d\xi} = - \frac{2\xi u_\infty}{h_\infty} \frac{du_\infty}{d\xi} (1 - IS_3) f'g$$

$$+ \frac{u_\infty^2}{h_\infty} \ell \left(\frac{\partial f'}{\partial \eta} \right)^2 + \frac{\partial}{\partial \eta} \left[\frac{\ell}{P_R} \frac{\partial g}{\partial \eta} \right] + \frac{T_{e_\infty}}{T_\infty} \frac{\partial}{\partial \eta} \left[\lambda \frac{\partial \theta}{\partial \eta} \right]$$

$$+ \frac{5}{2} \frac{\sqrt{2\xi}\, kT_{e_\infty} \alpha_1 j_{y_\infty}}{(\rho\mu)_r u_\infty C_p T_\infty e} \frac{\partial \theta}{\partial \eta} + \frac{5}{2} \frac{\sqrt{2\xi}\, kT_{e_\infty} \alpha_2 E_{x_\infty}}{(\rho\mu)_r u_\infty C_p T_\infty e} \frac{\partial \theta}{\partial \eta}$$

$$+ \frac{5}{2} \frac{\sqrt{2\xi}\, kT_{e_\infty} \alpha'_1 j_{y_\infty}}{(\rho\mu)_r u_\infty C_p T_\infty e} \theta + \frac{5}{2} \frac{\sqrt{2\xi}\, kT_{e_\infty} \alpha'_2 E_{x_\infty}}{(\rho\mu)_r u_\infty C_p T_\infty e} \theta$$

(Cont'd)

$$+\frac{2\xi}{C_p T_\infty}\,\frac{1}{(\rho\mu)_r\,\rho_\infty u_\infty^2}\left[\frac{\sigma E_{x_\infty}^2}{1+\beta_e\beta_i}+\frac{j_{y_\infty}^2}{\sigma}\;\frac{(1+\beta_e\beta_i)^2+\beta_e^2}{1+\beta_e\beta_i}\right] g$$

$$+\frac{2\xi B_z j_{y_\infty} f'g}{(\rho\mu)_r C_p T_\infty \rho_\infty u_\infty}\,(1-IS_3)-\frac{IS_1 T_{e_\infty}(2\xi)}{C_p T_\infty \rho_\infty}\,gf'\,\frac{\partial\theta}{\partial\xi}$$

$$-\frac{I S_1 T_{e_\infty} V}{\rho_\infty C_p T_\infty}\,g\frac{\partial\theta}{\partial\eta}-\frac{I S_1 (2\xi)\dfrac{dT_{e_\infty}}{d\xi}}{\rho_\infty C_p T_\infty}\,g f'\theta$$

$$+\frac{IS_2(2\xi)}{\rho_\infty}\,g f'\frac{\partial g}{\partial\xi}+\frac{IS_2 V}{\rho_\infty}\,g\frac{\partial g}{\partial\eta}+\frac{IS_2(2\xi)}{\rho_\infty T_\infty}\left(\frac{dT_\infty}{d\xi}\right) g^2 f'$$

$$-\frac{2\xi I n_e}{C_p T_\infty \rho_\infty}\,f'\,\frac{\partial g}{\partial\xi}-\frac{I n_e}{C_p T_\infty\rho_\infty}\,V\frac{\partial g}{\partial\eta}-\frac{I n_e(2\xi)}{C_p T_\infty^2\rho_\infty}\,\frac{dT_\infty}{d\xi}\,f'g$$

$$+\frac{I n_e j_{y_\infty}(2\xi)B_z}{p(\rho\mu)_r\rho_\infty C_p T_\infty u_\infty}\,g f'-\frac{I u_\infty n_e(2\xi)}{p C_p T_\infty}\,\frac{du_\infty}{d\xi}\,gf'$$

Electron energy:

$$\frac{3}{2}kn_e+\left[\frac{3}{2}kT_{e_\infty}\theta+I_{C_s}\right]S_1\left(\frac{(\rho\mu)_r T_{e_\infty}u_\infty^2}{2\xi}\right)\left[2\xi f'\frac{\partial\theta}{\partial\xi}\right.$$

$$\left.+V\frac{\partial\theta}{\partial\eta}+2\xi\frac{f'\theta}{T_{e_\infty}}\,\frac{dT_{e_\infty}}{d\xi}\right]-S_2\left[\frac{3}{2}k\,T_{e_\infty}\theta+I\right]\left(\frac{(\rho\mu)_r h_\infty u_\infty^2}{2\xi}\right)\left[2\xi f'\frac{\partial g}{\partial\xi}\right.$$

$$\left.+V\frac{\partial g}{\partial\eta}+2\xi\frac{f'g}{T_\infty}\,\frac{dT_\infty}{d\xi}\right]-\left(\frac{3}{2}kT_{e_\infty}\theta+I\right)S_3\rho_\infty u_\infty^3\,\frac{du_\infty}{d\xi}\,(\rho\mu)_r f'$$

$$+\left(\frac{3}{2}kT_{e_\infty}\theta+I\right)S_3\,j_{y_\infty}B_z u_\infty f'-\frac{\rho_\infty u_\infty^2 C_p(\rho\mu)_r T_{e_\infty}}{2\xi g}\,\frac{\partial\theta}{\partial\eta}\left(\lambda\frac{\partial\theta}{\partial\eta}\right)$$

(Cont'd)

$$- \frac{\rho_\infty u_\infty}{g\sqrt{2\xi}} \frac{5}{2} \frac{k}{e} T_{e_\infty} j_{y_\infty} \frac{\partial}{\partial \eta}(\theta\alpha_1) - \frac{\rho_\infty u_\infty}{g\sqrt{2\xi}} \frac{5}{2} \frac{k}{e} T_{e_\infty} E_{x_\infty} \frac{\partial}{\partial \eta}(\theta\alpha_2)$$

$$+ n_e \left[\frac{5}{2} k T_{e_\infty} \theta + I\right] \left[\frac{(\rho\mu)_r u_\infty^2}{2\xi\, g}\right] \left[2\xi f' \frac{\partial g}{\partial \xi} + V \frac{\partial g}{\partial \eta} - \frac{2\xi f' g}{\rho_\infty} \frac{d\rho_\infty}{d\xi}\right]$$

$$= E_{x_\infty}^2 \frac{\sigma}{(1+\beta_e\beta_i)^2} + \frac{j_{y_\infty}^2}{\sigma} \frac{\left[(1+\beta_e\beta_i)^2 + \beta_e^2\right]}{(1+\beta_e\beta_i)^2}$$

$$+ 3 m_e n_e k T_{e_\infty} \left(g \frac{T_\infty}{T_{e_\infty}} - \theta\right) \sum_s \frac{\nu_s}{m_s}$$

Finally, we reduce these equations to finite difference form and obtain the following results

Momentum eq.:

$$A_{11} f_{m+1,\,n+1} + B_{11} f'_{m+1,n} + B_{12}\, g_{m+1,n} + C_{11} f'_{m+1,n-1} = D_1$$

where

$$A_{11} = \frac{V\Delta\xi}{8\xi f'\Delta\eta} - \frac{\ell\Delta\xi}{4\xi f'(\Delta\eta)^2} - \frac{\ell'\Delta\xi}{8\xi f'}$$

$$B_{11} = 1 + \frac{\Delta\xi}{2u_\infty} \frac{du_\infty}{d\xi} + \frac{\ell\Delta\xi}{2\xi f'(\Delta\eta)^2}$$

$$B_{12} = - \frac{\Delta\xi}{2u_\infty f'} \frac{du_\infty}{d\xi}$$

$$C_{11} = - \frac{V\Delta\xi}{8\xi f'\Delta\eta} - \frac{\ell\,\Delta\xi}{4\xi f'(\Delta\eta)^2} + \frac{\ell'\Delta\xi}{8\xi f'(\Delta\eta)}$$

$$D_1 = f'_{m,n}\left(1 - \frac{\Delta\xi}{2u_\infty} \frac{du_\infty}{d\xi}\right) - \frac{V\Delta\xi}{4\xi f'} f'_\eta + \frac{\Delta\xi}{2u_\infty f} \frac{du_\infty}{d\xi} g_{m,n}$$

$$+ \frac{\ell\Delta\xi}{4\xi f'} f'_{\eta\eta} + \frac{\ell'(\Delta\xi)}{4\xi f'} f'_\eta$$

Overall Energy Equation

$$A_{21}\, f'_{m+1,n+1} + A_{22}\, g_{m+1,n+1} + A_{23}\, \theta_{m+1,n+1} + B_{21}\, f'_{m+1,n} + B_{22}\, g_{m+1,n} + B_{23}\, \theta_{m+1,n} + C_{21}\, f'_{m+1,n-1} + C_{22}\, g_{m+1,n-1} + C_{23}\, \theta_{m+1,n-1} = D_2$$

where

$$A_{21} = \frac{-\Delta\xi}{2\xi f'} \frac{u_\infty^2}{C_p T_\infty} \frac{\ell\, f'_n}{2\Delta\eta}$$

$$A_{22} = \frac{\Delta\xi}{8\xi f'\Delta\eta} \left[V - \frac{2\ell}{P_R \Delta\eta} - \left(\frac{\ell}{P_R}\right)' + \frac{I n_e V}{C_p T_\infty \rho_\infty} - \frac{I S_2 V}{\rho_\infty} \quad g_{m,n} \right]$$

$$A_{23} = \frac{\Delta\xi}{8\xi f'\Delta\eta} \left[-\frac{2\lambda}{\Delta\eta} \frac{T_{e_\infty}}{T_\infty} - \lambda' \frac{T_{e_\infty}}{T_\infty} \right.$$

$$\left. \frac{5\sqrt{2\xi}\, kT_\infty}{2(\rho\mu)_r u_\infty C_p T_\infty e} \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}\right) + \frac{I S_1 T_{e_\infty} V}{\rho_\infty C_p T_\infty} g_{m,n} \right]$$

$$B_{21} = 0$$

$$B_{22} = 1 - \frac{\Delta\xi}{2} \left\{ - \frac{u_\infty}{C_p T_\infty} \frac{du_\infty}{d\xi} (1 - I S_3) - \frac{1}{T_\infty} \frac{dT_\infty}{d\xi} \right.$$

$$+ \frac{1}{f' C_p T_\infty (\rho\mu)_r \rho_\infty u_\infty^2} \left[\frac{\sigma E_{x_\infty}^2}{1+\beta_e\beta_i} + \frac{j_{y_\infty}^2}{\sigma} \frac{(1+\beta_e\beta_i)^2 + \beta_e^2}{1+\beta_e\beta_i} \right]$$

$$+ \frac{B_z\, j_{y_\infty}}{(\rho\mu)_r C_p T_\infty \rho_\infty u_\infty} (1 - I S_3) - \frac{I n_e}{C_p T_\infty^2 \rho_\infty} \frac{dT_e}{d\xi}$$

$$\left. + \frac{I n_e\, j_{y_\infty} B_z}{p(\rho\mu)_2 \rho_\infty C_p T_\infty u_\infty} - \frac{I u_\infty n_e}{p C_p T_\infty} \frac{du_\infty}{d\xi} \right\} + \frac{I S_1 (\Delta\xi)}{\rho_\infty C_p T_\infty} \frac{dT_{e_\infty}}{d\xi} \theta_{m,n}$$

$$- \frac{I S_2}{\rho_\infty} g_{m,n} - \frac{(\Delta\xi)\, I S_2}{\rho_\infty T_\infty} \frac{dT_\infty}{d\xi} g_{m,n} + \frac{I n_e}{C_p \rho_\infty T_\infty} + \frac{\Delta\xi}{2\xi f'(\Delta\eta)^2} \frac{\ell}{P_R}$$

$$B_{23} = \frac{\Delta\xi}{2\xi f'(\Delta\eta)^2} \frac{T_{e_\infty}}{T_\infty} \lambda - \frac{\Delta\xi}{4\xi f'} \frac{5}{2} \frac{\sqrt{2\xi}\, kT_{e_\infty}}{(\rho\mu)_2 u_\infty C_p T_\infty e} \left(\alpha_1' j_{y_\infty} + \alpha_2' E_{x_\infty}\right)$$

$$+ \frac{IS_1 T_{e_\infty}}{C_p T_\infty \rho_\infty} g_{m,n} + \frac{IS_1(\Delta\xi)}{\rho_\infty C_p T_\infty} \frac{dT_{e_\infty}}{d\xi} g_{m,n}$$

$$C_{21} = \frac{\Delta\xi}{4\xi f'\Delta\eta} \frac{u_\infty^2}{C_p T_\infty} \ell\, f'_\eta = -A_{21}$$

$$C_{22} = \frac{\Delta\xi}{8\xi f'\Delta\eta} \left[-V - \frac{2}{\Delta\eta}\left(\frac{\ell}{P_R}\right) + \left(\frac{\ell}{P_R}\right)' + \frac{IS_2 V}{\rho_\infty} g_{m,n} - \frac{In_e V}{C_p T_\infty \rho_\infty} \right]$$

$$C_{23} = \frac{\Delta\xi}{8\xi f'\Delta\eta} \left[-2\frac{T_{e_\infty}}{T_\infty} \frac{\lambda}{\Delta\eta} + \frac{T_{e_\infty}}{T_\infty} \lambda' + \frac{5}{2} \frac{\sqrt{2\xi}\, k\, T_\infty}{(\rho\mu)_r u_\infty C_p T_\infty e} \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}\right) \right.$$

$$\left. - \frac{IS_1 T_{e_\infty} V}{\rho_\infty C_p T_\infty} g_{m,n} \right]$$

$$D_2 = -\frac{\Delta\xi}{4\xi f'} V g_\eta + \frac{1}{2} g_{m,n} \left[2 - \frac{(\Delta\xi) u_\infty}{C_p T_\infty} \frac{du_\infty}{d\xi} (1 - IS_3) - \frac{(\Delta\xi)}{T_\infty} \frac{dT_\infty}{d\xi} \right.$$

$$+ \frac{\Delta\xi}{f' C_p T_\infty (\rho\mu)_r \rho_\infty u_\infty^2} \left(\frac{\sigma E_{x_\infty}^2}{1+\beta_e\beta_i} + \frac{j_{y_\infty}^2}{\sigma} \frac{(1+\beta_e\beta_i)^2 + \beta_e^2}{1+\beta_e\beta_i} \right)$$

$$+ \frac{(\Delta\xi) B_z j_{y_\infty}}{(\rho\mu)_r C_p T_\infty \rho_\infty u_\infty} (1 - IS_3) - \frac{In_e(\Delta\xi)}{C_p T_\infty^2 \rho_\infty} \frac{dT_\infty}{d\xi} + \frac{In_e(\Delta\xi)\, j_{y_\infty} B_z}{p(\rho\mu)_r \rho_\infty C_p T_\infty u_\infty}$$

$$\left. - \frac{I u_\infty n_e(\Delta\xi)}{p\, C_p T_\infty} \frac{du_\infty}{d\xi} \right] + \frac{\Delta\xi}{4\xi f'} \left[\frac{\ell}{P_R} g_{\eta\eta} + \left(\frac{\ell}{P_R}\right)' g_\eta \right.$$

$$\left. + \frac{T_{e_\infty}}{T_\infty} \lambda\, \theta_{\eta\eta} + \frac{T_{e_\infty}}{T_\infty} \lambda' \theta_\eta + \frac{5}{2} \frac{\sqrt{2\xi}\, k T_\infty}{(\rho\mu)_r u_\infty C_p T_\infty e} \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty}\right) \theta_\eta \right]$$

$$+\frac{5}{2}\frac{\sqrt{2\xi}\;kT_{e_\infty}}{(\rho\mu)_r u_\infty C_p T_\infty e}\left(\alpha_1' j_{y_\infty}+\alpha_2' E_{x_\infty}\right)\theta_{m,n}-\frac{IS_1 T_{e_\infty} V}{\rho_\infty C_p T_\infty}\, g_{m,n}\theta_\eta$$

$$\left.+\frac{IS_2 V}{\rho_\infty}\, g_{m,n}\, g_\eta-\frac{In_e V}{C_p T_\infty \rho_\infty}\, g_\eta\right]$$

$$+\frac{IS_1 T_{e_\infty}}{C_p T_\infty \rho_\infty}\, g_{m,n}\,\theta_{m,n}-\frac{IS_2}{\rho_\infty}\, g^2_{m,n}+\frac{In_e}{C_p T_\infty \rho_\infty}\, g_{m,n}$$

Electron Energy Equation

$$A_{31}\, f'_{m+1,n+1}+A_{32}\, g_{m+1,n+1}+A_{33}\,\theta_{m+1,n+1}+B_{31}\, f'_{m+1,n}$$

$$+B_{32}\, g_{m+1,n}+B_{33}\,\theta_{m+1,n}+C_{31}\, f'_{m+1,n-1}+C_{32}\, g_{m+1,n-1}$$

$$+C_{33}\,\theta_{m+1,n-1}=D_3$$

where

$$A_{31}=0$$

$$A_{32}=\left[\frac{n_e\left(\frac{5}{2}kT_{e_\infty}\theta+I\right)}{g}-\left(\frac{3}{2}kT_{e_\infty}\theta+I\right)S_2 C_p T_\infty\right]\frac{(\rho\mu)_r u_\infty^2 V}{8\xi(\Delta\eta)}$$

$$A_{33}=\left[\frac{3}{2}\,k n_e+\left(\frac{3}{2}kT_{e_\infty}\,\theta+I\right)S_1\right]\frac{(\rho\mu)_r T_{e_\infty} u_\infty^2 V}{8\xi\,\Delta\eta}$$

$$-\frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty}\lambda}{4\xi g\,(\Delta\eta)^2}-\frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty}\lambda'}{8\xi g\,\Delta\eta}$$

$$-\frac{5k\rho_\infty u_\infty T_{e_\infty}}{8eg\sqrt{2\xi}\,\Delta\eta}\left(\alpha_1 j_{y_\infty}+\alpha_2 E_{x_\infty}\right)$$

$$B_{31}=\left[\frac{3}{2}kn_e+\left(\frac{3}{2}kT_{e_\infty}\,\theta+I\right)S_1\right]\frac{(\rho\mu)_r u_\infty^2}{2}\,\frac{dT_{e_\infty}}{d\xi}\,\theta_{m,n}$$

$$-n_e\left(\frac{5}{2}kT_{e_\infty}\,\theta+I\right)\frac{(\rho\mu)_r u_\infty^2}{2}\,\frac{1}{\rho_\infty}\,\frac{d\rho_\infty}{d\xi}$$

$$- S_2\left(\frac{3}{2} k T_{e_\infty} \theta + I\right) \frac{(\rho\mu)_r u_\infty^2}{2} C_p \frac{dT_\infty}{d\xi} g_{m,n}$$

$$- \frac{3}{4} k T_{e_\infty} S_3 (\rho\mu)_r \rho_\infty u_\infty^3 \frac{du_\infty}{d\xi} \theta_{m,n} - \frac{I S_3}{2} (\rho\mu)_r \rho_\infty u_\infty^3 \frac{du_\infty}{d\xi}$$

$$+ \frac{3}{4} k T_{e_\infty} S_3 u_\infty j_{y_\infty} B_z \theta_{m,n} + \frac{I S_3 u_\infty}{2} j_{y_\infty} B_z$$

$$B_{32} = \left[\frac{n_e\left(\frac{5}{2} k T_{e_\infty} \theta + I\right)}{g} - \left(\frac{3}{2} k T_{e_\infty} \theta + I\right) S_2 C_p T_\infty \right] (\rho\mu)_r u_\infty^2 \frac{f'_{m,n}}{\Delta\xi}$$

$$- S_2\left(\frac{3}{2} k T_{e_\infty} \theta + I\right) \frac{(\rho\mu)_r u_\infty^2}{2} C_p \frac{dT_\infty}{d\xi} f'_{m,n}$$

$$- \frac{3}{2} m_e n_e k T_\infty \sum_s \frac{\nu_{e_s}}{m_s}$$

$$B_{33} = \left[\frac{3}{2} k n_e + \left(\frac{3}{2} k T_{e_\infty} \theta + I\right) S_1\right] \frac{(\rho\mu)_r T_{e_\infty} u_\infty^2 f'}{\Delta\xi}$$

$$+ \left[\frac{3}{2} k n_e + \left(\frac{3}{2} k T_{e_\infty} \theta + I\right) S_1\right] \frac{(\rho\mu)_r u_\infty^2}{2} \frac{dT_{e_\infty}}{d\xi} f'_{m,n}$$

$$- \frac{3}{4} k T_{e_\infty} S_3 (\rho\mu)_r \rho_\infty u_\infty^3 \frac{du_\infty}{d\xi} f'_{m,n}$$

$$+ \frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty} \lambda}{2\xi g (\Delta\eta)^2} - \frac{\rho_\infty u_\infty T_{e_\infty}}{\sqrt{2\xi}\; g} \frac{5k}{4e}\left(\alpha_1' j_{y_\infty} + \alpha_2' E_{x_\infty}\right)$$

$$+ \frac{3}{2} m_e n_e k T_{e_\infty} \sum_s \frac{\nu_{e_s}}{m_s} + \frac{3}{4} k T_{e_\infty} S_3 u_\infty B_z f'_{m,n}$$

$$C_{31} = 0$$

$$C_{32} = - \left[\frac{n_e\left(\frac{5}{2} k T_{e_\infty} \theta + I\right)}{g} - \left(\frac{3}{2} k T_{e_\infty} \theta + I\right) S_2 C_p T_\infty \right] \frac{(\rho\mu)_r u_\infty^2 V}{8\xi \Delta\eta}$$

$$C_{33} = - \left[\frac{3}{2}\, k\, n_e + \left(\frac{3}{2} k\, T_{e_\infty} \theta + I \right) S_1 \right] \frac{(\rho\mu)_r T_{e_\infty} u_\infty^2 V}{8\xi (\Delta\eta)}$$

$$- \frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty} \lambda}{4\xi g (\Delta\eta)^2} + \frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty} \lambda'}{8\xi g (\Delta\eta)}$$

$$+ \frac{\rho_\infty u_\infty T_{e_\infty}}{2\sqrt{2\xi}\, g(\Delta\eta)} \frac{5k}{4e} \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty} \right)$$

$$D_3 = \left[\frac{3}{2} k\, n_e + \left(\frac{3}{2} k\, T_{e_\infty} \theta + I \right) S_1 \right] \frac{(\rho\mu)_r T_{e_\infty} u_\infty^2 f'}{\Delta\xi} \theta_{m,n}$$

$$- \left[\frac{3}{2} k\, n_e + \left(\frac{3}{2} k T_{e_\infty} \theta + I \right) S_1 \right] \frac{(\rho\mu)_r T_{e_\infty} u_\infty^2 V}{4\xi} \theta_\eta$$

$$+ \left[n_e \frac{\left(\frac{5}{2} k\, T_{e_\infty} \theta + I \right)}{g} - \left(\frac{3}{2} k T_{e_\infty} \theta + I \right) S_2 C_p T_\infty \right] (\rho\mu)_r u_\infty^2 \frac{f'_{m,n}}{\Delta\xi} g_{m,n}$$

$$- \left[n_e \frac{\left(\frac{5}{2} k\, T_{e_\infty} \theta + I \right)}{g} - \left(\frac{3}{2} k\, T_{e_\infty} \theta + I \right) S_2 C_p T_\infty \right] \frac{(\rho\mu)_r u_\infty^2 V}{4\xi} g_\eta$$

$$+ n_e \left(\frac{5}{2}\, k T_{e_\infty} \theta + I \right) (\rho\mu)_r u_\infty^2 \frac{1}{\rho_\infty} \frac{d\rho_\infty}{d\xi} \frac{f'_{m,n}}{2}$$

$$+ \frac{IS_3}{2} (\rho\mu)_r \rho_\infty u_\infty^3 \frac{du_\infty}{d\xi} f'_{m,n} + \frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty} \lambda}{4\xi g} \theta_{\eta\eta}$$

$$+ \frac{(\rho\mu)_r C_p \rho_\infty u_\infty^2 T_{e_\infty} \lambda'}{4\xi g} \theta_\eta + \frac{\rho_\infty u_\infty T_{e_\infty}}{\sqrt{2\xi}\, g} \frac{5k}{4e} \left(\alpha_1 j_{y_\infty} + \alpha_2 E_{x_\infty} \right) \theta_\eta$$

$$+\frac{\rho_\infty u_\infty T_{e_\infty}}{\sqrt{2\xi}\; g}\frac{5k}{4e}\left(\alpha_1' j_{y_\infty}+\alpha_2' E_{x_\infty}\right)\theta_{m,n}+\frac{\sigma}{(1+\beta_\ell\beta_i)^2}E_{x_\infty}^{\;2}$$

$$+\frac{j_{y_\infty}^{\;2}}{\sigma}\frac{\left[(1+\beta_e\beta_i)^2+\beta_e^{\;2}\right]}{(1+\beta_e\beta_i)^2}+\frac{3}{2}m_e n_e kT_\infty\sum_s\frac{\nu_{es}}{m_s}g_{m,n}$$

$$-\frac{3}{2}m_e n_e kT_{e_\infty}\hat{\sum_s}\frac{\nu_{es}}{m_s}\theta_{m,n}-IS_3\frac{u_\infty j_{y_\infty}}{2}B_z\quad f'_{m,n}$$

The boundary conditions must then be specified. We have as a result of channel flow calculations (carried out by Mr. Nichols group at Lewis Laboratories) the several variables at the outer edge of the boundary layer.

$$u_\infty = u_\infty(x)$$

$$p_\infty = p_\infty(x)$$

$$T_\infty = T_\infty(x)$$

$$T_{e_\infty} = T_{e_\infty}(x)$$

Along the wall we have

$$u = v = 0$$

$$T = T_w(x) \quad \text{or} \quad q = q_w(x)$$

Finally, the boundary condition on T_e is obtained by equating the normal current flow from the wall to that which passes through the sheath, and requiring continuity of heat flux across the sheath interface. Thus we have

$$j_{y_\infty}+i_w=\frac{n_{e_w} e\langle V_e\rangle}{4}\,e^{-\frac{e\Delta\varphi}{RT_{e_w}}}{}_v-n_{e_w}e\sqrt{\frac{kT_{e_w}}{m_c}}$$

and

$$K_e\left(\frac{\partial T_e}{\partial y}\right)_w + \frac{5}{2}\,\frac{j_{e_y}}{e}\,kT_{e_w} = (2kT_{e_w} + e|\Delta\varphi|)\frac{n_{e_w}\langle V_e\rangle}{4}\,e^{-\frac{e|\Delta\varphi|}{kT_{e_w}}}$$

To put this into suitable finite difference form then we proceed as follows:

At the wall we want a linear B.C. to fit in with our linear system of finite difference equations. Thus,

$$w_1 = Hw_2 + Fw_3 + h \qquad w = \begin{Bmatrix} f' \\ g \\ \theta \end{Bmatrix}_{m+1,n}$$

We note that $f' = 0$ at the wall (n=1). Also, T_{wall} = const. so $q_1 = T_w/T_\infty(\xi) = g_w(\xi)$. For θ we have problems. We have two equations to work with, each containing θ and $\Delta\varphi$, and have to eliminate $\Delta\varphi$ between them. So

$$j_{y_\infty} + i_w = \frac{n_{e_w} e\langle V_e\rangle}{4}\,e^{\frac{-e|\Delta\varphi|}{kT_{e_\infty}\theta_w}} - n_{e_w} e\sqrt{\frac{kT_{e_\infty}\theta_w}{m_c}}$$

and

$$K_e T_{e_\infty}\left(\frac{\partial \theta}{\partial y}\right)_w + \frac{5}{2}\left(j_{e_y}\right)_w \frac{k}{e}\,T_{e_\infty}\theta_w = \left(2kT_{e_\infty}\theta_w + e|\Delta\varphi|\right)\times$$

$$\times\frac{n_{e_w}\langle V_e\rangle}{4}\,e^{\frac{-e|\Delta\varphi|}{kT_{e_\infty}\theta_w}}$$

alternately,

$$j_{y_\infty} + i_w = \frac{\dfrac{\lambda C_p(\rho\mu)_r u_\infty}{k\sqrt{2\xi}}\left(\dfrac{\partial\theta}{\partial\eta}\right)_w + \dfrac{5}{2}\left(j_{e_y}\right)_w\dfrac{\theta_w}{e}}{2\theta_w + \dfrac{e|\Delta\varphi|}{kT_{e_\infty}}} - n_{e_w} e\sqrt{\frac{kT_{e_\infty}\theta_w}{m_c}}$$

Next write

$$\left.\frac{\partial \theta}{\partial \eta}\right|_w = \frac{-3\theta_1+4\theta_2-\theta_3}{2\Delta\eta}$$

Then we have

$$j_{y_\infty} + i_w = \frac{\frac{\lambda C_p(\rho\mu)_r u_\infty}{k\sqrt{2\xi}}\left[\frac{-3\theta_1+4\theta_2-\theta_3}{2\Delta\eta}\right] + \frac{5}{2}\left(j_{e_y}\right)_w \frac{\theta_1}{e}}{2\theta_1 + \frac{e|\Delta\varphi|}{kT_{e_\infty}}}$$

$$- n_{e_w} e \sqrt{\frac{kT_{e_\infty}\theta_1}{m_c}}$$

or

$$\left(j_{y_\infty} + i_w\right)\left(2\theta_1 + \frac{e|\Delta\varphi|}{kT_{e_\infty}}\right) = \frac{\lambda C_p(\rho\mu)_r u_\infty}{k\sqrt{2\xi}}\frac{\left(-3\theta_1+4\theta_2-\theta_3\right)}{2\Delta\eta} + \frac{5}{2}\left(j_{e_y}\right)_w \frac{\theta_1}{e}$$

$$-en_{e_w}\sqrt{\frac{kT_{e_\infty}\theta_1}{m_c}}\left[2\theta_1 + \frac{e|\Delta\varphi|}{kT_{e_\infty}}\right]$$

We find $|\Delta\varphi|$ from the 1st of our original two equations

$$e^{-\frac{e|\Delta\varphi|}{kT_{e_\infty}\theta_1}} = \left[\left(j_{y_\infty} + i_w\right) + en_{e_w}\sqrt{\frac{kT_{e_\infty}\theta_1}{m_c}}\right]\frac{4}{en_{e_w}}\sqrt{\frac{\pi m_c}{8kT_{e_\infty}\theta_1}}$$

$$= \left[\frac{\left(j_{y_\infty} + i_w\right)}{en_{e_w}}\sqrt{\frac{2\pi m_e}{kT_{e_\infty}\theta_1}} + \sqrt{\frac{2\pi m_e}{m_c}}\right] = a^{-1}$$

$$\frac{e|\Delta\varphi|}{kT_{e_\infty}\theta_1} = \ln a^{-1} = \ln a \quad \text{or} \quad \frac{e|\Delta\varphi|}{kT_{e_\infty}} = \theta_1 \ln a$$

so,

$$\theta_1 \left\{ \left(j_{y_\infty} + i_w\right)\left(2 + \ln a\right) + \frac{3\lambda C_p(\rho\mu)_r u_\infty}{2k\sqrt{2\xi}\;\Delta\eta} - \frac{5}{2}\left(j_{e_y}\right)_w \right.$$

$$\left. + en_{e_w}\sqrt{\frac{kT_{e_\infty}\theta_1}{m_c}}\;(2 + \ln a) \right\} = \left(\frac{2\lambda C_p(\rho\mu)_r u_\infty}{k\sqrt{2\xi}\;\Delta\eta}\right)\theta_2$$

$$- \left(\frac{\lambda C_p(\rho\mu)_r u_\infty}{2k\sqrt{2\xi}\;\Delta\eta}\right)\theta_3$$

This is highly non-linear in θ_1. Since we wish to deal with a linear system treat θ_1 in $\sqrt{}$ and "a" terms as $\theta_{m,1}$ while θ_1, θ_2, θ_3 in linear terms are $\theta_{m+1,1}$ etc. Actually, we must express our boundary condition at $m + 1/2$, n. So,

$$\frac{\theta_{m+1,1} + \theta_{m,1}}{2} = B_2\left[\frac{\theta_{m+1,2} + \theta_{m,2}}{2}\right] + B_3\left[\frac{\theta_{m+1,3} + \theta_{m,3}}{2}\right]$$

where

$$B_2 = \frac{\dfrac{2\lambda_1 C_p(\rho\mu)_r u_\infty}{k\sqrt{2\xi}\;(\Delta\eta)}}{\left[2 + \ln a\right]\left[j_{y_\infty} + i_w + en_{e_w}\sqrt{\dfrac{kT_{e_\infty}\theta_1}{m_c}}\right] + \dfrac{3\lambda_1 C_p(\rho\mu)_r u_\infty}{2k\sqrt{2\xi}\;\Delta\eta} - \dfrac{5}{2}\left(j_{e_y}\right)_w}$$

$$B_3 = -\frac{1}{4}B_2$$

and where

$$a = \left[\frac{\left(j_{y_\infty} + i_w\right)}{en_{e_w}}\sqrt{\frac{2\pi m_e}{kT_{e_\infty}\theta_1}} + \sqrt{\frac{2\pi m_e}{m_c}}\right]^{-1}$$

$$n_{e_w} = n_e(g_1, \theta_1) \quad ; \quad (j_{e_y})_w = j_{e_y}(g_1, \theta_1)$$

$$\lambda_1 = \lambda(g_1, \theta_1)$$

Also all ξ dependent quantities, j_{y_∞}, u_∞, T_{e_∞}, are evaluated at $m + 1/2$ location.

Now θ_1 will be treated as an iterable quantity. That is, for first calculation take $\theta_{m,1}$, and for subsequent iterations take $\frac{\theta_{m,1} + \theta_{m+1,1}}{2}$.

Finally,

$$\theta_{m+1,1} = B_2\theta_{m+1,2} + B_3\theta_{m+1,3} + \left[B_2\theta_{m,2} + B_3\theta_{m,3} - \theta_{m,1}\right]$$

or

$$\theta_{m+1,1} = B_2\theta_{m+1,2} + B_3\theta_{m+1,3} + B_4$$

Then

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_2 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3 \end{bmatrix}$$

$$h = \begin{Bmatrix} 0 \\ g_1 \\ B_4 \end{Bmatrix}$$

where $g_1 = T_w / T_{\infty m+1}$ and T_w = constant, $T_{\infty m+1}$ corresponds to $T_\infty(\xi)$ evaluated when ξ corresponds to $m + 1$ location.

Note that $B_4 = 0$ in first calculation but could be non-zero for any subsequent iterations!

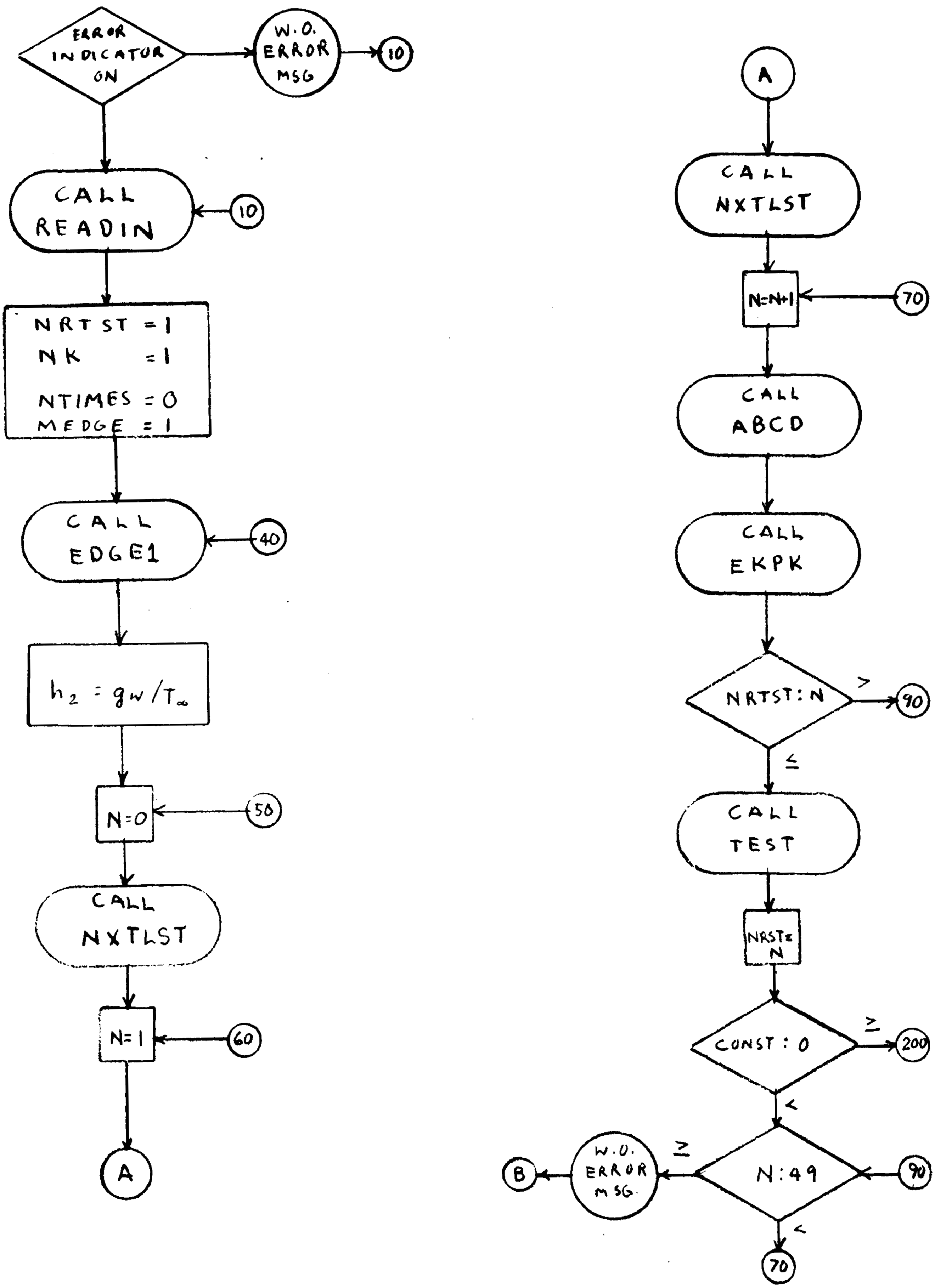
III. PROGRAMMING FOR COMPUTER CALCULATIONS

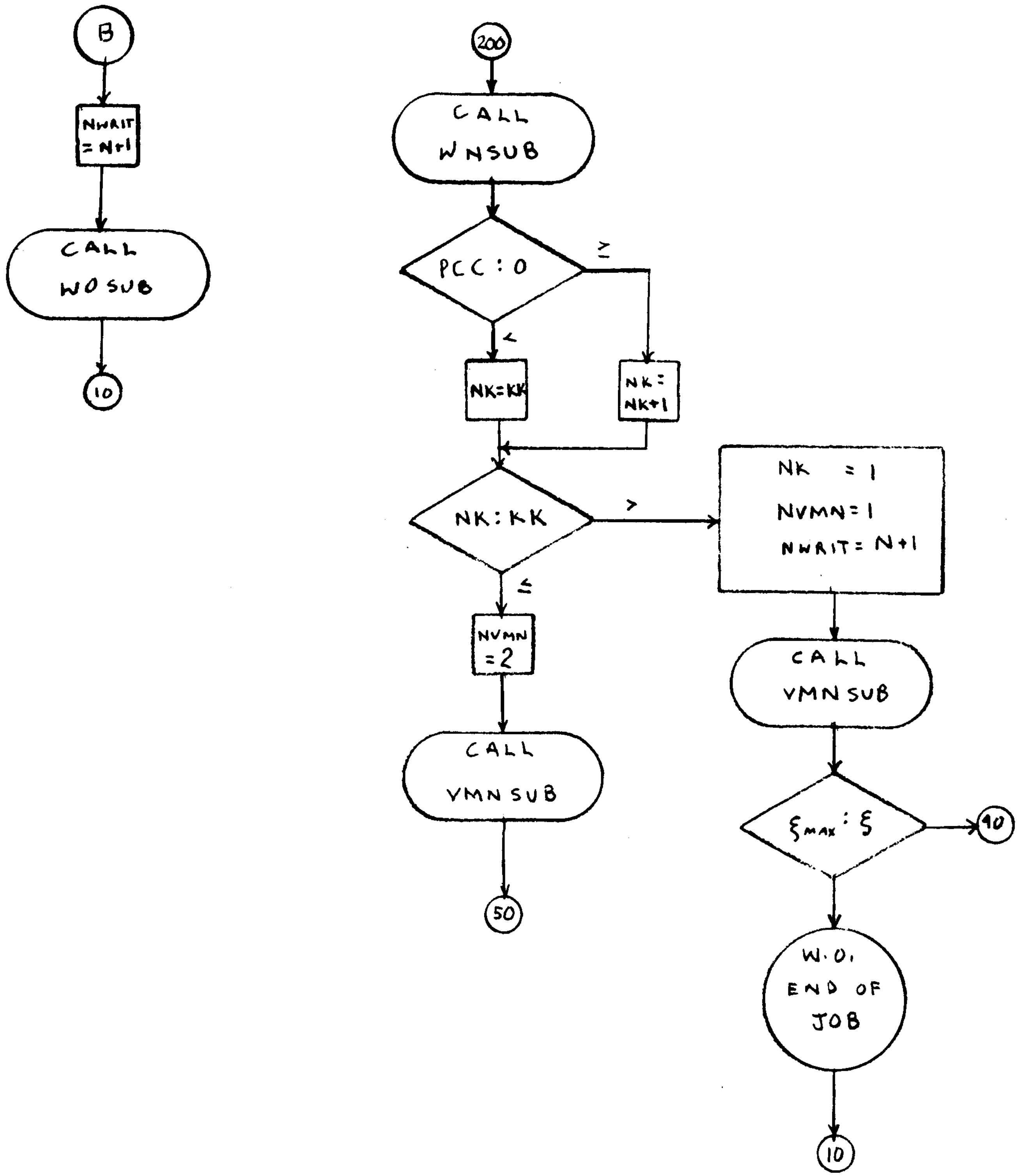
The flow charts for the computer program developed for numerical solution of the boundary layer equations according to an implicit finite difference scheme are given in the following pages.

In summary, the dependent variables from a given profile are assumed known at the M^{th} station. Use of these & the free-stream quantities known at every station allows then, through the finite difference scheme, the determination of these quantities of profiles at the M+1 station, the next point along the channel. The program basically proceeds as follows:

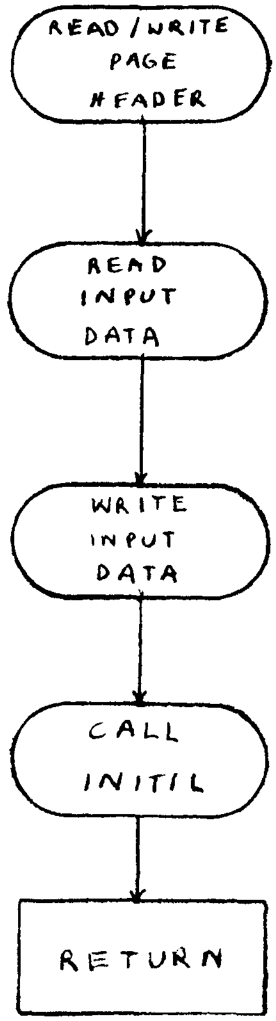
1) Initial conditions are set at the M^{th} station, and profiles are assumed at the M+1 station.
2) Average profiles are computed for the $M + \frac{1}{2}$ station.
3) The coefficients of the difference equation are computed based on the average profiles.
4) New profiles are solved for at the M+1 station from the difference equation.
5) Comparison of the new profile at the M+1 station to any previously computed profiles at the M+1 station is made and a convergence test is performed.
6) If convergence is established, the profile is written out and the next station is treated; if convergence is not satisfied, the most recent profile is used in repeating from step (2). The iterative process for determining the profile at the M+1 station may be terminated either by satisfaction of the convergence test, or completion of a prescribed number of iterations.

This program has been written for a more general case for some work of Blottner. It has been successfully run, and the computation scheme is known to be valid. The major effort up to this time has been expended in acquiring a familiararity with this extremely complex general program and extracting from it those sequents immediately applicable to the present task. A very efficient program has evolved from this procedure with the additional benefit of having a guideline to follow, in the form of the general program, which will facilitate debugging.

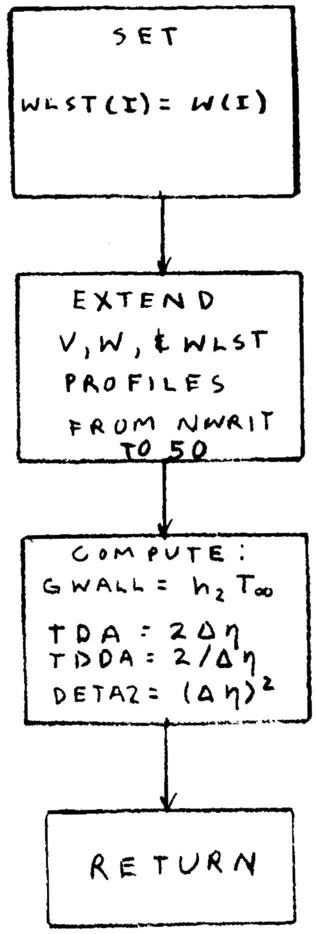

ERROR INDICATOR ON
W.O. ERROR MSG
10
CALL READIN
10
NRTST = 1
NK = 1
NTIMES = 0
MEDGE = 1
CALL EDGE1
40
$h_2 = g_w/T_\infty$
N=0
50
CALL NXTLST
N=1
60
A
A
CALL NXTLST
N=N+1
70
CALL ABCD
CALL EKPK
NRTST:N
>
90
≤
CALL TEST
NRST= N
CONST : 0
≥
200
<
N:49
90
≥
W.O. ERROR MSG.
B
<
70


B
NWRIT = N+1
CALL WOSUB
10
200
CALL WNSUB
PCC : 0
≥
<
NK=KK
NK= NK+1
NK : KK
>
≤
NK = 1
NVMN = 1
NWRIT = N+1
NVMN = 2
CALL VMNSUB
CALL VMNSUB
50
ξmax : ξ
40
W.O. END OF JOB
10

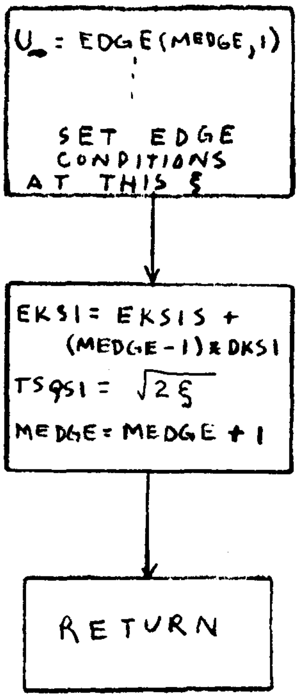
SUBROUTINE READIN



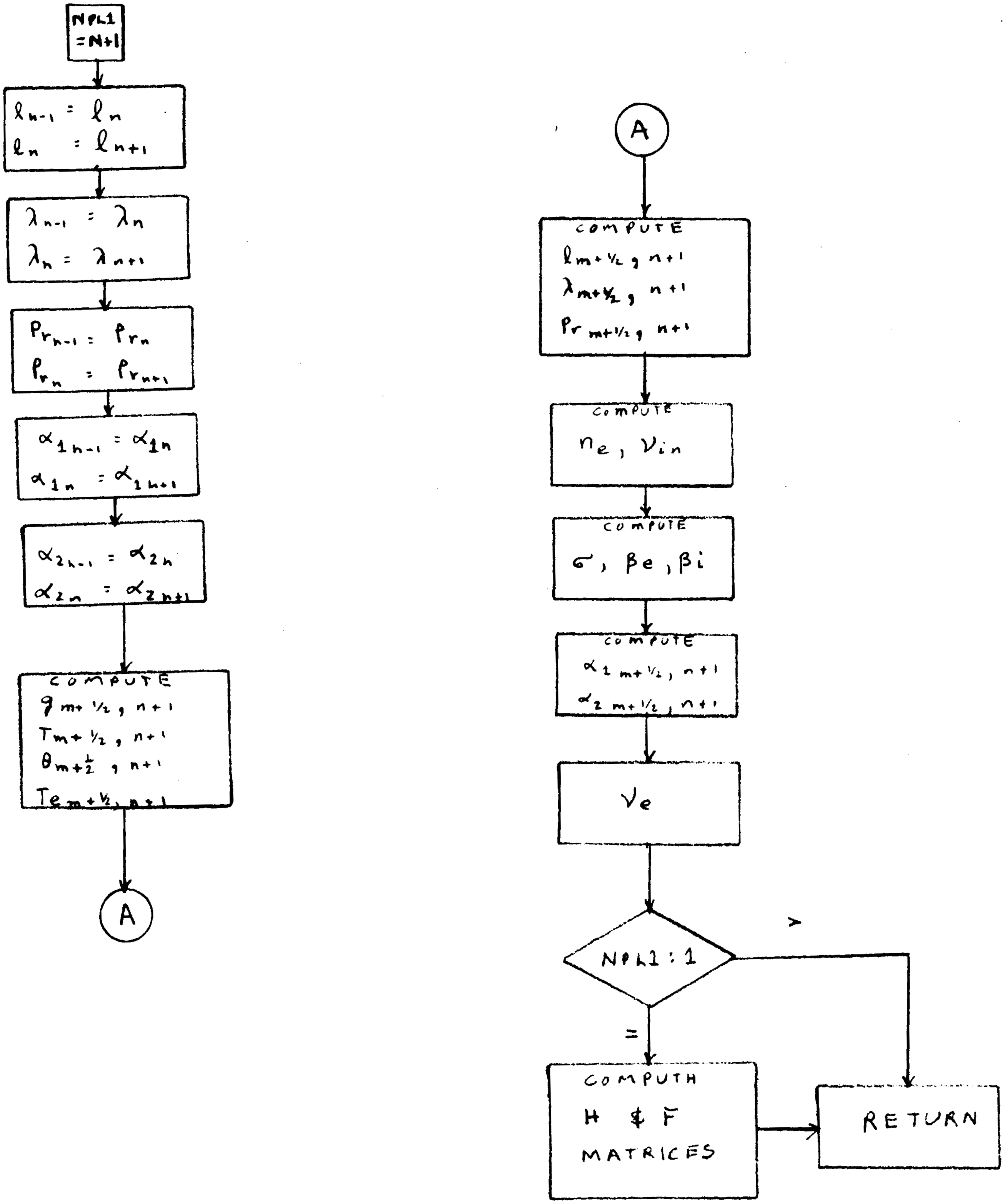
SUBROUTINE INITIAL



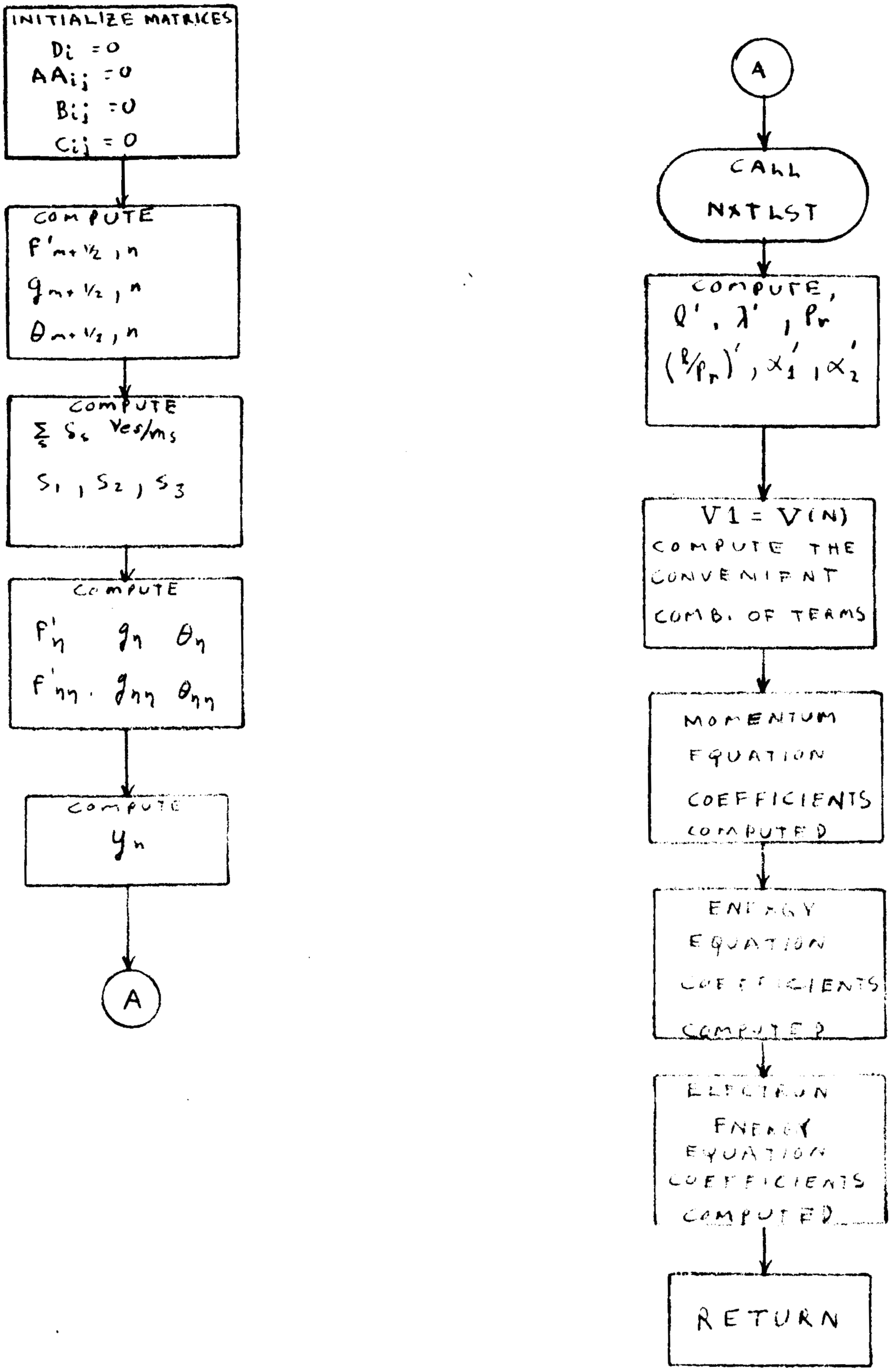
SUBROUTINE EDGE 1



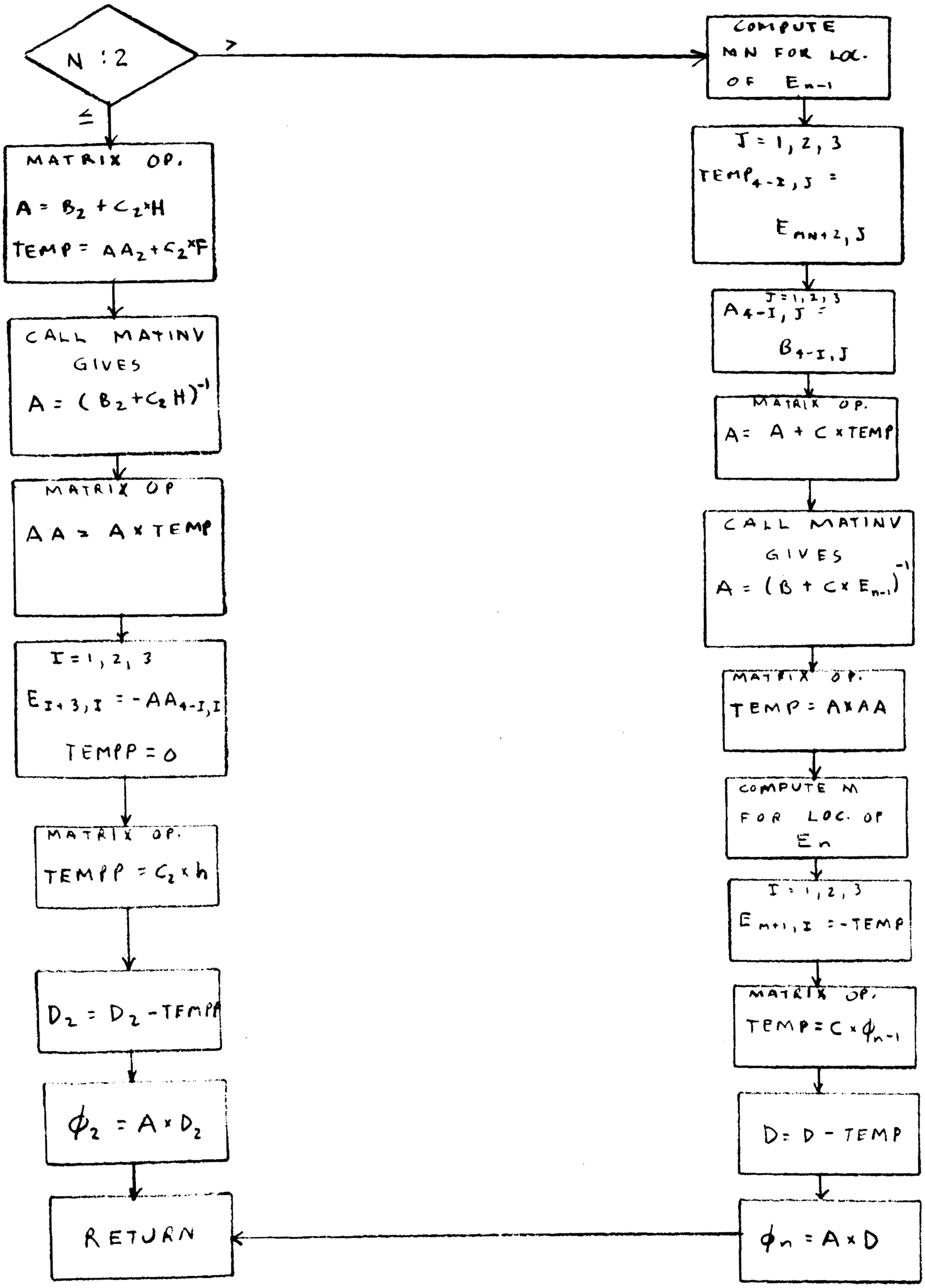
SUBROUTINE NXTLST



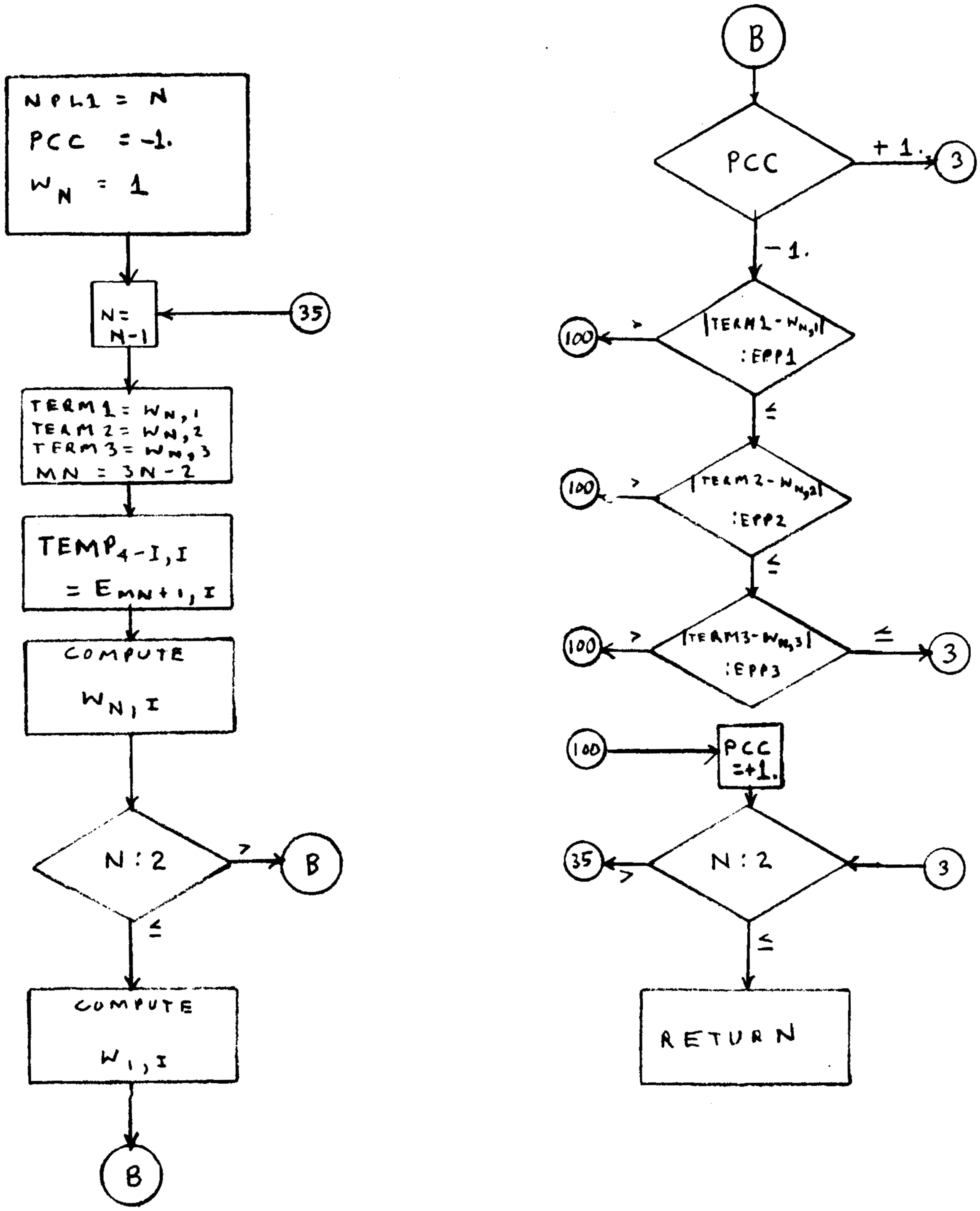
SUBROUTINE ABCD



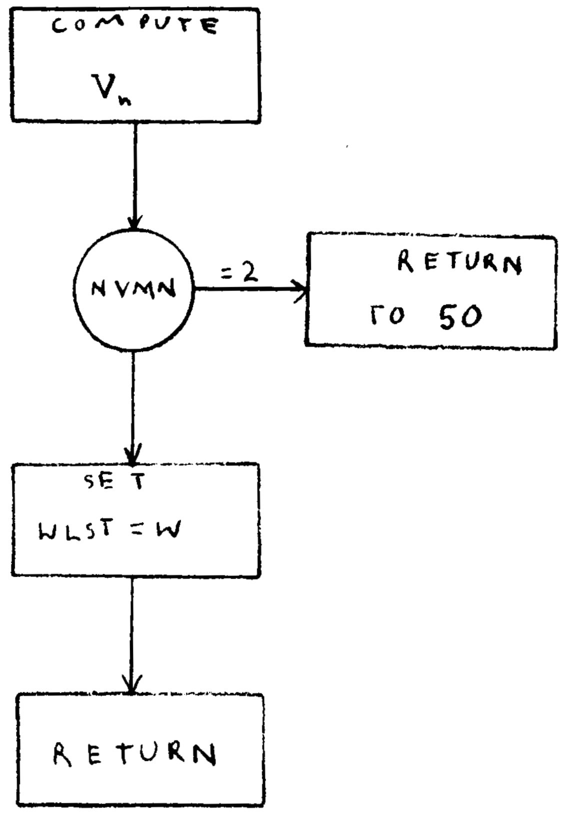
SUBROUTINE EKPK



SUBROUTINE WNSUB



SUBROUTINE VMNSUB



IV. FURTHER WORK

During the first quarter major emphasis was placed on bringing a computor program into existence. This required an entensive reduction of the governing equations, which has now been completed. The numerical techniques are reasonably in hand, the program has been flow charted, and all subroutine programs have been written in Fortran IV.

During the second quarter a punch card source deck will be typed, compilation will be carried out and debugging will proceed. One particular channel flow will be used for initial calculations. We expect to use the $M = 0.5$ case recently calculated by Nichols for this purpose.

As we proceed to evolve this program, further theoretical work will be carried out to better our understanding of the electron energy equation, the necessary transport properties, and the sheath description. It is hoped that interpretation of the calculated results will proceed alongside the above theoretical studies.